On the mystery of quantum probabilistic rule: trigonometric and hyperbolic probabilistic behaviours

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Abstract

We demonstrate that the origin of so called quantum probabilistic rule (which differs from the classical Bayes' formula by the presence of $\cos\theta$-factor) might be explained in the framework of ensemble fluctuations which are induced by preparation procedures. In particular, quantum rule for probabilities (with nontrivial $\cos\theta$-factor) could be simulated for macroscopic physical systems via preparation procedures producing ensemble fluctuations of a special form. We discuss preparation and measurement procedures which may produce probabilistic rules which are neither classical nor quantum; in particular, hyperbolic 'quantum theory.'

1 Introduction

It is well known that the classical probabilistic rule based on the Bayes' formula for conditional probabilities cannot be applied to quantum formalism, see, for example, [1]-[3] for extended discussions. In fact, all special properties of quantum systems are just consequences of violations of the classical probability rule, Bayes' theorem [1]. In this paper we restrict our investigations to the two dimensional case. Here Bayes' formula has the form $(i = 1, 2)$:

$$\mathbf{p}(A = a_i) = \mathbf{p}(C = c_1)\mathbf{p}(A = a_i/C = c_1) + \mathbf{p}(C = c_2)\mathbf{p}(A = a_i/C = c_2), \tag{1}$$

where A and C are physical variables which take, respectively, values a_1, a_2 and c_1, c_2. Symbols $\mathbf{p}(A = a_i/C = c_j)$ denote conditional probabilities. There is a large diversity of opinions on the origin of violations of (1) in quantum mechanics. The common opinion is that violations of (1) are induced by special properties of quantum systems.

Let ϕ be a quantum state. Let $\{\phi_i\}_{i=1}^2$ be an orthogonal basis consisting of eigenvectors of the operator $\hat{C}$ corresponding to the physical observable C. The quantum theoretical rule has the form $(i = 1, 2)$:

$$q_i = \mathbf{p}_1\mathbf{p}_{1i} + \mathbf{p}_2\mathbf{p}_{2i} \pm 2\sqrt{\mathbf{p}_1\mathbf{p}_{1i}\mathbf{p}_2\mathbf{p}_{2i}}\cos\theta, \tag{2}$$

where $q_i = \mathbf{p}_\phi(A = a_i), \mathbf{p}_j = \mathbf{p}_\phi(C = c_j), \mathbf{p}_{ij} = \mathbf{p}_{\phi_i}(A = a_j), i, j = 1, 2.$ Here probabilities have indexes corresponding to quantum states. The common opinion is that this quantum probabilistic rule must be considered as a peculiarity of nature. However, there exists an opposition to this general opinion, namely the probabilistic opposition. The main domain of activity of this probabilistic opposition is Bell's inequality and the EPR paradox [4] , see, for example, [1], [5]-[11]. The general idea supported by the probabilistic opposition is that special quantum behaviour can be understood on the basis of local realism, if we be careful with the probabilistic description of physical phenomena. It seems that the origin of all 'quantum troubles' is probabilistic rule (2). It seems that the violation of Bell's inequality is just a new representation of the old contradiction between rules (1) and (2) (the papers of Accardi [1] and De Muynck, De Baere and Martens [7] contain extended discussions on this problem). Therefore, the main problem of the probabilistic justification of quantum mechanics is to find the clear probabilistic explanation of the origin of quantum probabilistic rule (2) and the violation of classical probabilistic rule (1) and explain why (2) is sometimes reduced to (1).

L. Accardi [5] introduced a notion of the *statistical invariant* to investigate the relation between classical Kolmogorovean and quantum probabilistic models, see also Gudder and Zanghi in [6]. He was also the first who mentioned that Bayes' postulate is a "hidden axiom of the Kolmogorovean model... which limits its applicability to the statistical description of the natural phenomena ", [5]. In fact, this investigation plays a crucial role in our analysis of classical and quantum probabilistic rules.

An interesting investigation on this problem is contained in the paper of J. Shummhammer [11]. He supports the idea that quantum probabilistic rule (2) is not a peculiarity of nature, but just a consequence of one special method of the probabilistic description of nature, so called method of *maximum predictive power*. We do not directly support the idea of Shummhammer. It seems that the origin of (2) is not only a consequence of the use of one special method for the description of nature, but merely a consequence of our manipulations with nature, ensembles of physical systems, in quantum preparation/measurement procedures.

In this paper we provide probabilistic analysis of quantum rule (2). In our analysis 'probability' has the meaning of the *frequency probability,* namely the limit of frequencies in a long sequence of trials (or for a large statistical ensemble). Hence, in fact , we follow to R. von Mises' approach to probability [12]. It seems that it would be impossible to find the roots of quantum rule (2) in the conventional probability framework, A. N. Kolmorogov, 1933, [13]. In the conventional measure-theoretical framework probabilities are defined as sets of real numbers having some special mathematical properties. Classical rule (1) is merely a consequence of the definition of conditional probabilities. In the Kolmogorov framework to analyse the transition from (1) to (2) is to analyse the transition from one definition to another. In the frequency framework we can analyse behaviour of trails which induce one or another property of probability. Our analysis shows that quantum probabilistic rule (2) can be explained on the basis of ensemble fluctuations (one of possible sourses of ensemble fluctuations is so called ensemble nonreproducibility, see De Baere [7]; see also [10] for the statistical variant of nonreproducibility). Such fluctuations can generate (under special conditions) the $\cos\theta$-factor in (2). Thus trigonometric fluctuations of quantum probabilities can be explained without using the wave arguments.

An unexpected consequence of our analysis is that quantum probability rule (2) is just one of possible perturbations (by ensemble fluctuations) of classical probability rule (1). In principle, there might exist experiments which would produce perturbations of classical probabilistic rule (1) which differ from quantum probabilistic rule (2).

2 Quantum formalism and ensemble fluctuations

1. Frequency probability theory. The frequency definition of probability is more or less standard in quantum theory; especially in the approach based on preparation and measurement procedures, [14], [3].

Let us consider a sequence of physical systems $\pi = (\pi_1, \pi_2, ..., \pi_N, ...)$. Suppose that elements of π have some property, for example, position, and this property can be described by natural numbers: $L = \{1, 2, ..., m\}$, the set of labels. Thus, for each $\pi_j \in \pi$, we have a number $x_j \in L$. So π induces a sequence

$$x = (x_1, x_2, ..., x_N, ...), \quad x_j \in L. \tag{3}$$

For each fixed $\alpha \in L$, we have the relative frequency $\nu_N(\alpha) = n_N(\alpha)/N$ of the appearance of α in $(x_1, x_2, ..., x_N)$. Here $n_N(\alpha)$ is the number of elements in $(x_1, x_2, ..., x_N)$ with $x_j = \alpha$. R. von Mises [12] said that x satisfies to the principle of the *statistical stabilization* of relative frequencies, if, for each fixed $\alpha \in L$, there exists the limit

$$\mathbf{p}(\alpha) = \lim_{N \to \infty} \nu_N(\alpha). \tag{4}$$

This limit is said to be a probability of α.

We shall not consider so called principle of *randomness,* see [12] for the details. This principle, despite its importance for the foundations of probability theory, is not related to our frequency analysis. We shall be interested only in the statistical stabilization of relative frequencies.

2. Preparation and measurement procedures and quantum formalism. We consider a statistical ensemble S of quantum particles described by a quantum state ϕ. This ensemble is produced by some preparation procedure $\mathcal{E}$, see, for example, [14], [3] for details. There are two discrete physical observables $C = c_1, c_2$ and $A = a_1, a_2$.

The total number of particles in S is equal to N. Suppose that $n_i^c, i = 1, 2,$ particles in S would give the result $C = c_i$ and $n_i^a, i = 1, 2,$ particles in S would give the result $A = a_i$. Suppose that, among those particles which would produce $C = c_i$, there are $n_{ij}, i, j, = 1, 2,$ particles which would give the result $A = a_j$ (see (R) and (C) below to specify the meaning of 'would give'). So

$n_i^c = n_{i1} + n_{i2}, n_j^a = n_{1j} + n_{2j}, i, j = 1, 2.$

(R) We can use an objective realist model in that both C and A are *objective properties* of a quantum particle, see [2], [3], [10] for the details. In such a model we can consider in S sub-ensembles $\mathrm{S_j(C)}$ and $\mathrm{S_j(A), j = 1, 2}$, of particles having properties $\mathrm{C = c_j}$ and $A = a_j$, respectively. Set $\mathrm{S_{ij}(A, C) = S_i(C) \cap S_j(A)}$. Then n_{ij} is the number of elements in the ensemble $\mathrm{S_{ij}(A, C)}$. We remark that the 'existence' of the objective property ($C = c_i$ and $A = a_j$) need not imply the possibility to measure this property. For example, such a measurement is impossible in the case of incompatible observables. So in general ($C = c_i$ and $A = a_j$) is a kind of hidden property.

(C) We can use so called *contextualist* realism, see, for example, [3] and De Muynck, De Baere and Martens in [7]. Here we cannot assume that a quantum system determines uniquely the result of a measurement. This result depends not only on properties of a quantum particle, but also on the experimental arrangement. Here n_{ij} is the number of particles which would produce $C = c_i$ and $A = a_j$. We remark that the latter statement, in fact, contains *counterfactuals*: $C = c_i$ and $A = a_j$ could not be measured simultaneously, see, for example, [3] for the use of counterfactuals in quantum theory.

The quantum experience says that the following frequency probabilities are well defined for all observables C, A :

$$\mathbf{p}_i = \mathbf{p}_\phi(C = c_i) = \lim_{N \to \infty} \mathbf{p}_i^{(N)}, \mathbf{p}_i^{(N)} = \frac{n_i^c}{N}; \tag{5}$$

$$q_i = \mathbf{p}_\phi(A = a_i) = \lim_{N \to \infty} q_i^{(N)}, q_i^{(N)} = \frac{n_i^a}{N}. \tag{6}$$

Can we say something about behaviour of frequencies $\tilde{\mathbf{p}}_{\mathrm{ij}}^{\mathrm{(N)}} = \frac{\mathrm{n_{ij}}}{\mathrm{N}}, \mathrm{N} \to \infty$? Suppose that they stabilize, when $\mathrm{N} \to \infty$. This implies that probabilities $\tilde{\mathbf{p}}_{\mathrm{ij}} = \mathbf{p}_\phi(\mathrm{C = c_i, A = a_j}) = \lim_{N \to \infty} \tilde{\mathbf{p}}_{\mathrm{ij}}^{\mathrm{(N)}}$ would be well defined. The quantum experience says that (in general) this is not the case. Thus, in general, the frequencies $\tilde{\mathbf{p}}_{ij}^{(N)}$ fluctuate, when $\mathrm{N} \to \infty$. Such fluctuations can, nevertheless, produce the statistical stabilization (5), (6), see [10] for the details.

Remark 2.1. The common interpretation of experimental violations of Bell's inequality is that realism and even contextualist realism cannot be used in quantum theory (at least in the local framework). However, Bell's considerations only imply that we cannot use realist models under the assumption that $\tilde{\mathbf{p}}_{\mathrm{ij}}^{\mathrm{(N)}}$ stabilize. The realist models with fluctuating frequencies $\tilde{\mathbf{p}}_{\mathrm{ij}}^{\mathrm{(N)}}$ can coexist with violations of Bell's inequality, see [10].

Let us now consider statistical ensembles $T_i, i = 1, 2,$ of quantum particles described by the quantum states ϕ_i which are eigenstates of the operator $\hat{C}$: $\hat{C}\phi_i = c_i\phi_i$. These ensembles are produced by some preparation procedures $\mathcal{E}_i$. For instance, we can suppose that particles produced by a preparation procedure $\mathcal{E}$ for the quantum state ϕ pass through additional filters $F_i, i = 1, 2$. In quantum formalism we have

$$\phi = \sqrt{\mathbf{p}_1}\ \phi_1 + \sqrt{\mathbf{p}_2} e^{i\theta}\ \phi_2\ . \tag{7}$$

In the objective realist model (R) this representation may induce the illusion that ensembles $T_i, i = 1, 2,$ for states ϕ_i must be identified with sub-ensembles $\mathrm{S_i(C)}$ of the ensemble S for the state ϕ. However, there are no physical reasons for such an identification. There are two main sources of troubles with this identification:

(a). The additional filter $\mathrm{F_1}$ (and $\mathrm{F_2}$) changes the properties of quantum particles. The probability distribution of the property A for the ensemble $\mathrm{S_1(C)} = \{\pi \in \mathrm{S} : \mathrm{C}(\pi) = \mathrm{c_1}\}$ (and $\mathrm{S_2(C)}$) may differ from the corresponding probability distribution for the ensemble $\mathrm{T_1}$(and $\mathrm{T_2}$). So different preparation procedures produce different distributions of properties. The same conclusion can be done for the contextualist realism: an additional filter changes possible reactions of quantum particles to measurement devices.

(b). As we have already mentioned, frequencies $\tilde{\mathbf{p}}_{\mathrm{ij}}^{(\mathrm{N})} = \frac{\mathrm{n_{ij}}}{\mathrm{N}}$ must fluctuate (in the case of incompatible observables). Therefore, even if additional filters do not change properties of quantum particles, nonreproducibility implies that the distribution of the property A may be essentially different for statistical ensembles $\mathrm{S_1(C)}$ and $\mathrm{S_2(C)}$ (sub-ensembles of S) and $\mathrm{T_1}$ and $\mathrm{T_2}$. Moreover, distributions may be different even for sub-ensembles $\mathrm{S_1(C)}$ and $\mathrm{S'_1(C)}$ (or $\mathrm{S_2(C)}$ and $\mathrm{S'_2(C)}$), of two different ensembles S and S$'$ of quantum particles prepared in the same quantum state ϕ, see [10].

Fluctuations of physical properties which could be induced by (a) or (b) will be called *ensemble fluctuations.*

Suppose that $\mathrm{m_{ij}}$ particles in the ensemble $\mathrm{T_i}$ would produce the result $A = a_j, j = 1, 2$. We can use the objective realist model, (R). Then m_{ij} is just the number of particles in the ensemble T_i having the objective property $A = a_j$. We can also use the contextualist model, (C). Then m_{ij} is the number of particles in the ensemble T_i which in the process of an interaction with a measurement device for the physical observable A would give the result $A = a_j$.

The quantum experience says that the following frequency probabilities are well defined:

$\mathbf{p}_{\rm ij}=\mathbf{p}_{\phi_{\rm i}}({\rm A}={\rm a_j})=\lim_{{\rm N}\to\infty}\mathbf{p}^{({\rm N})}_{\rm ij}, \mathbf{p}^{({\rm N})}_{\rm ij}=\frac{{\rm m_{ij}}}{{\rm n^c_i}}$.

Here it is assumed that an ensemble $\rm T_i$ consists of $\rm n^c_i$ particles, $i=1,2$. It is also assumed that $\rm n^c_i=n^c_i(N)\to\infty, N\to\infty$. In fact, the latter assumption holds true if both probabilities $\mathbf{p}_i, i=1,2$, are nonzero.

We remark that probabilities $\mathbf{p}_{ij}=\mathbf{p}_{\phi_i}(A=a_j)$ cannot be (in general) identified with conditional probabilities $\mathbf{p}_\phi(A=a_j/C=c_i)=\frac{\tilde{\mathbf{p}}_{ij}}{\mathbf{p}_i}$. As we have remarked, these probabilities are related to statistical ensembles prepared by different preparation procedures, namely by $\mathcal{E}_i, i=1,2$, and $\mathcal{E}$.

Let $\{\psi_j\}_{j=1}^2$ be an orthonormal basis consisting of eigenvectors of the operator A. We can restrict our considerations to the case:

$$\phi_1=\sqrt{\mathbf{p}_{11}}\;\psi_1+e^{i\gamma_1}\sqrt{\mathbf{p}_{12}}\;\psi_2,\;\;\phi_2=\sqrt{\mathbf{p}_{21}}\;\psi_1+e^{i\gamma_2}\sqrt{\mathbf{p}_{22}}\;\psi_2\;. \tag{8}$$

As $(\phi_1,\phi_2)=0$, we obtain:

$\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}+{\rm e}^{{\rm i}(\gamma_1-\gamma_2)}\sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}=0$.

Hence, $\sin(\gamma_1-\gamma_2)=0$ (we suppose that all probabilities $\mathbf{p}_{ij}>0$) and $\gamma_2=\gamma_1+\pi k$. We also have

$\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}+\cos(\gamma_1-\gamma_2)\sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}=0$.

This implies that $k=2l+1$ and $\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}=\sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}$. As $\mathbf{p}_{12}=1-\mathbf{p}_{11}$ and $\mathbf{p}_{21}=1-\mathbf{p}_{22}$, we obtain that

$$\mathbf{p}_{11}=\mathbf{p}_{22},\;\;\mathbf{p}_{12}=\mathbf{p}_{21}. \tag{9}$$

This equalities are equivalent to the condition: $\mathbf{p}_{11}+\mathbf{p}_{21}=1, \mathbf{p}_{12}+\mathbf{p}_{22}=1$. So the matrix of probabilities $(\mathbf{p}_{ij})_{i,j=1}^2$ is so called *double stochastic matrix,* see, for example, [3] for general considerations.

Thus, in fact,

$$\phi_1=\sqrt{\mathbf{p}_{11}}\;\psi_1+e^{i\gamma_1}\sqrt{\mathbf{p}_{12}}\;\psi_2,\;\phi_2=\sqrt{\mathbf{p}_{21}}\;\psi_1-e^{i\gamma_1}\sqrt{\mathbf{p}_{22}}\;\psi_2. \tag{10}$$

So $\varphi={\rm d}_1\psi_1+{\rm d}_2\psi_2$, where

$d_1=\sqrt{\mathbf{p}_1\mathbf{p}_{11}}+e^{i\theta}\sqrt{\mathbf{p}_2\mathbf{p}_{21}},\;d_2=e^{i\gamma_1}\sqrt{\mathbf{p}_1\mathbf{p}_{12}}-e^{i(\gamma_1+\theta)}\sqrt{\mathbf{p}_2\mathbf{p}_{22}}$.

Thus

$$q_1=\mathbf{p}_\phi(A=a_1)=|d_1|^2=\mathbf{p}_1\mathbf{p}_{11}+\mathbf{p}_2\mathbf{p}_{21}+2\sqrt{\mathbf{p}_1\mathbf{p}_{11}\mathbf{p}_2\mathbf{p}_{21}}\cos\theta; \tag{11}$$

$$q_2=\mathbf{p}_\phi(A=a_2)=|d_2|^2=\mathbf{p}_1\mathbf{p}_{12}+\mathbf{p}_2\mathbf{p}_{22}-2\sqrt{\mathbf{p}_1\mathbf{p}_{12}\mathbf{p}_2\mathbf{p}_{22}}\cos\theta. \tag{12}$$

3. Probablity relations connecting preparation procedures. Let us forget at the moment about the quantum theory. We consider an arbitrary preparation procedure $\mathcal{E}$ for microsystems or macrosystems. Suppose that $\mathcal{E}$ produced an ensemble S of physical systems. Let $C(= c_1, c_2)$ and $A(= a_1, a_2)$ be physical quantities which can be measured for elements $\pi \in S$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be preparation procedures which are based on filters F_1 and F_2 corresponding, respectively, to values c_1 and c_2 of C. Denote statistical ensembles produced by these preparation procedures by symbols T_1 and T_2, respectively. Symbols $N, n_i^c, n_i^a, n_{ij}, m_{ij}$ have the same meaning as in the previous considerations. Probablities $\mathbf{p}_i, \mathbf{p}_{ij}, q_i$ are defined in the same way as in the previous considerations. The only difference is that, instead of indexes corresponding to quantum states, we use indexes corresponding to statistical ensembles: $\mathbf{p}_i = \mathbf{P}_S(C = c_i), q_i = \mathbf{P}_S(A = a_i), \mathbf{p}_{ij} = \mathbf{P}_{T_i}(A = a_i)$.

In the classical frequency framework we obtain:

$$q_1^{(N)} = \frac{n_1^a}{N} = \frac{n_{11}}{N} + \frac{n_{21}}{N} = \frac{m_{11}}{N} + \frac{m_{21}}{N} + \frac{(n_{11} - m_{11})}{N} + \frac{(n_{21} - m_{21})}{N}.$$

But, for $i = 1, 2$, we have
$\frac{m_{1i}}{N} = \frac{m_{1i}}{n_1^c} \cdot \frac{n_1^c}{N} = \mathbf{p}_{1i}^{(N)} \mathbf{p}_1^{(N)}, \quad \frac{m_{2i}}{N} = \frac{m_{2i}}{n_2^c} \cdot \frac{n_2^c}{N} = \mathbf{p}_{2i}^{(N)} \mathbf{p}_2^{(N)}$.
Hence

$$q_i^{(N)} = \mathbf{p}_1^{(N)} \mathbf{p}_{1i}^{(N)} + \mathbf{p}_2^{(N)} \mathbf{p}_{2i}^{(N)} + \delta_i^{(N)}, \tag{13}$$

where

$$\delta_i^{(N)} = \frac{1}{N} [(n_{1i} - m_{1i}) + (n_{2i} - m_{2i})], \ i = 1, 2.$$

In fact, this rest term depends on the statistical ensembles S, T_1, T_2, $\delta_i^{(N)} = \delta_i^{(N)}(S, T_1, T_2)$.

4. Behaviour of fluctuations. First we remark that $\lim_{N\to\infty} \delta_i^{(N)}$ exists for all physical measurements. This is a consequence of the property of statistical stabilization of relative frequencies for physical observables (in classical as well as in quantum physics). It may be that this property is a peculiarity of nature. It may be that this is just a property of our measurement and preparation procedures, see [10] for an extended discussion. In any case we always observe that
$\mathrm{q_i^{(N)}} \to \mathrm{q_i}, \mathbf{p}_\mathrm{i}^{(\mathrm{N})} \to \mathbf{p}_\mathrm{i}, \mathbf{p}_\mathrm{ij}^{(\mathrm{N})} \to \mathbf{p}_\mathrm{ij}, \mathrm{N} \to \infty$.
Thus there exist limits
$\delta_i = \lim_{N\to\infty} \delta_i^{(N)} = q_i - \mathbf{p}_1 \mathbf{p}_{1i} - \mathbf{p}_2 \mathbf{p}_{2i}$.

Suppose that ensemble fluctuations produce negligibly small (with respect to N) changes in properties of particles. Then

$$\delta_i^{(N)} \to 0, N \to \infty. \tag{14}$$

This asymptotic implies classical probablistic rule (1). In particular, this rule appears in all experiments of classical physics. Hence, preparation and measurement procedures of classical physics produce ensemble fluctuations with asymptotic (14). We also have such a behaviour in the case of compatible observables in quantum physics. Moreover, the same classical probabilistic rule we can obtain for incompatible observables C and A if the phase factor $\theta = \frac{\pi}{2} + \pi k$. Therefore classical probabilistic rule (1) is not directly related to commutativity of corresponding operators in quantum theory. It is a consequence of asymptotic (14) for ensemble fluctuations.

Suppose now that filters $F_i, i = 1, 2$, produce relatively large (with respect to N) changes in properties of particles. Then

$$\lim_{N\to\infty} \delta_i^{(N)} = \delta_i \neq 0. \tag{15}$$

Here we obtain probabilistic rules which differ from the classical one, (1). In particular, this implies that behaviour of ensemble fluctuations (15) cannot be produced in experiments of classical physics. A rather special class of ensemble fluctuations (15) is produced in experiments of quantum physics. However, ensemble fluctuations of form (15) are not reduced to quantum fluctuations (see further considerations).

To study carefully behaviour of fluctuations $\delta_i^{(N)}$, we represent them as: $\delta_i^{(N)} = 2\sqrt{\mathbf{p}_1^{(N)}\mathbf{p}_{1i}^{(N)}\mathbf{p}_2^{(N)}\mathbf{p}_{2i}^{(N)}}\lambda_i^{(N)}$,

where

$\lambda_i^{(N)} = \frac{1}{2\sqrt{m_{1i}m_{2i}}}[(n_{1i} - m_{1i}) + (n_{2i} - m_{2i})]$.

We have used the fact:

$\mathbf{p}_1^{(N)}\mathbf{p}_{1i}^{(N)}\mathbf{p}_2^{(N)}\mathbf{p}_{2i}^{(N)} = \frac{n_1^c}{N} \cdot \frac{m_{1i}}{n_1^c} \cdot \frac{n_2^c}{N} \cdot \frac{m_{2i}}{n_2^c} = \frac{m_{1i}m_{2i}}{N^2}$.

We have: $\delta_i = 2\sqrt{\mathbf{p}_1\mathbf{p}_{1i}\mathbf{p}_2\mathbf{p}_{2i}}\ \lambda_i$, where the coefficients $\lambda_i = \lim_{N\to\infty} \lambda_i^{(N)}, i = 1, 2$.

In classical physics the coefficients $\lambda_{\rm i} = 0$. The same situation we have in quantum physics for all compatible observables as well as for some incompatible observables. In the general case in quantum physics we can only say that

$$|\lambda_i| \leq 1. \tag{16}$$

Hence, for quantum fluctuations, we always have:

$$|\frac{(n_{1i}-m_{1i})+(n_{2i}-m_{2i})}{2\sqrt{m_{1i}m_{2i}}}| \leq 1, N \to \infty.$$

Thus quantum ensemble fluctuations induce a relatively small (but in general nonzero!) variations of properties.

4. Fluctuations which induce the quantum probabilistic rule. Let us consider preparation procedures $\mathcal{E}, \mathcal{E}_j, j=1,2$, which have the deviations, when $N \to \infty$, of the following form $(i=1,2)$:

$$\epsilon_{1i}^{(N)} = n_{1i} - m_{1i} = 2\xi_{1i}^{(N)}\sqrt{m_{1i}m_{2i}}, \tag{17}$$

$$\epsilon_{2i}^{(N)} = n_{2i} - m_{2j} = 2\xi_{2i}^{(N)}\sqrt{m_{1i}m_{2i}}, \tag{18}$$

where the coefficients ξ_{ij} satisfy the inequality

$$|\xi_{1i}^{(N)} + \xi_{2i}^{(N)}| \leq 1, N \to \infty. \tag{19}$$

Suppose that $\lambda_{\rm i}^{(\rm N)} = \xi_{1\rm i}^{(\rm N)} + \xi_{2\rm i}^{(\rm N)} \to \lambda_{\rm i}, \rm N \to \infty$, where $|\lambda_{\rm i}| \leq 1$. We can represent $\lambda_{\rm i}^{(\rm N)} = \cos\theta_{\rm i}^{(\rm N)}$. Then $\theta_i^{(N)} \to \theta_i, \mathrm{mod} 2\pi$, when $N \to \infty$. Thus $\lambda_{\rm i} = \cos\theta_{\rm i}$.

We obtained that:

$$\delta_i = 2\sqrt{\mathbf{p}_1\mathbf{p}_{1i}\mathbf{p}_2\mathbf{p}_{2i}}\cos\theta_i, i=1,2. \tag{20}$$

Thus fluctuations of the form (17), (18) produce the probability rule $(i = 1,2)$:

$$q_i = \mathbf{p}_1\mathbf{p}_{1i} + \mathbf{p}_2\mathbf{p}_{2i} + 2\sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{1i}\mathbf{p}_{2i}}\cos\theta_i. \tag{21}$$

The usual probabilistic calculations give us
$1 = q_1 + q_2 = \mathbf{p}_1\mathbf{p}_{11} + \mathbf{p}_2\mathbf{p}_{21} + +\mathbf{p}_1\mathbf{p}_{12} + \mathbf{p}_2\mathbf{p}_{22} +$
$2\ \sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{11}\mathbf{p}_{21}}\cos\theta_1 + 2\ \sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{12}\mathbf{p}_{22}}\cos\theta_2$
$= 1 + 2\sqrt{\mathbf{p}_1\mathbf{p}_2}[\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}\cos\theta_1 + \sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}\cos\theta_2]$.
Thus we obtain the relation:

$$\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}\cos\theta_1 + \sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}\cos\theta_2 = 0\ . \tag{22}$$

Suppose that ensemble fluctuations (17), (18), satisfy the additional condition

$$\lim_{N\to\infty}\mathbf{p}_{11}^{(N)} = \lim_{N\to\infty}\mathbf{p}_{22}^{(N)}. \tag{23}$$

This condition implies that the matrix of probabilities is a double stochastic matrix. Hence, we get

$$\cos\theta_1 = -\cos\theta_2 \; . \tag{24}$$

So we demonstrated that ensemble fluctuations (17), (18) in the combination with double stochastic condition (23) produce quantum probabilistic relations (11), (12).

It must be noticed that the existence of the limits $\lambda_i = \lim_{N\to\infty}\lambda_i^{(N)}$ does not imply the existence of limits $\xi_{1i} = \lim_{N\to\infty}\xi_{1i}^{(N)}$ and $\xi_{2i} = \lim_{N\to\infty}\xi_{2i}^{(N)}$. For example, let $\xi_{1i}^{(N)} = \lambda_i\cos^2\alpha_i^{(N)}$ and $\xi_{2i}^{(N)} = \lambda_i\sin^2\alpha_i^{(N)}$, where 'phases' $\alpha_i^{(N)}$ fluctuate mod 2π. Then numbers $\xi_{1\mathrm{i}}$ and $\xi_{2\mathrm{i}}$ are not defined, but $\lim_{\mathrm{N}\to\infty}[\xi_{1\mathrm{i}}^{(\mathrm{N})} + \xi_{2\mathrm{i}}^{(\mathrm{N})}] = \lambda_\mathrm{i}, \mathrm{i} = 1, 2,$ exist.

If $\xi_{ij}^{(N)}$ stabilize, then probabilities for the simultaneous measurement of incompatible observables would be well defined:

$\mathbf{p}(A = a_1, C = c_1) = \lim_{N\to\infty}\frac{n_{11}}{N} = \mathbf{p}_1\mathbf{p}_{11} + 2\sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{11}\mathbf{p}_{21}}\xi_{11}, \ldots.$

The quantum formalism implies that in general such probabilities do not exist.

Remark 2.1. The magnitude of fluctuations can be found experimentally. Let C and A be two physical observables. We prepare free statistical ensembles $\mathrm{S}, \mathrm{T}_1, \mathrm{T}_2$ corresponding to states ϕ, ϕ_1, ϕ_2. By measurements of C and A for $\pi \in \mathrm{S}$ we obtain frequencies $\mathbf{p}_1^{(\mathrm{N})}, \mathbf{p}_2^{(\mathrm{N})}, \mathrm{q}_1^{(\mathrm{N})}, \mathrm{q}_2^{(\mathrm{N})}$, by measurements of A for $\pi \in \mathrm{T}_1$ and for $\pi \in \mathrm{T}_2$ we obtain frequencies $\mathbf{p}_{1\mathrm{i}}^{(\mathrm{N})}$. We have

$\mathrm{f_i(N)} = \lambda_\mathrm{i}^{(\mathrm{N})} = \frac{\mathrm{q}_\mathrm{i}^{(\mathrm{N})} - \mathbf{p}_1^{(\mathrm{N})}\mathbf{p}_{1\mathrm{i}}^{(\mathrm{N})} - \mathbf{p}_2^{(\mathrm{N})}\mathbf{p}_{2\mathrm{i}}^{(\mathrm{N})}}{2\sqrt{\mathbf{p}_1^{(\mathrm{N})}\mathbf{p}_{1\mathrm{i}}^{(\mathrm{N})}\mathbf{p}_2^{(\mathrm{N})}\mathbf{p}_{2\mathrm{i}}^{(\mathrm{N})}}}$

It would be interesting to obtain graphs of functions $\mathrm{f_i(N)}$ for different pairs of physical observables. Of course, we know that $\lim_{N\to\infty} f_i(N) = \pm\cos\theta$. However, it may be that such graphs can present a finer structure of quantum states.

3 On the magnitude of fluctuations which produce the classical probabilistic rule

We remark that the classical probabilistic rule (which is induced by ensemble fluctuations with $\xi_i^{(N)} \to 0$) can be observed for fluctuations having relatively large absolute magnitudes. For instance, let

$$\epsilon_{1i}^{(N)} = 2\xi_{1i}^{(N)}\sqrt{m_{1i}}, \;\; \epsilon_{2i}^{(N)} = 2\xi_{2i}^{(N)}\sqrt{m_{2i}}, i = 1, 2, \tag{25}$$

where sequences of coefficients $\{\xi_{1i}^{(N)}\}$ and $\{\xi_{2i}^{(N)}\}$ are bounded ($N \to \infty$). Here

$\lambda_i^{(N)} = \frac{\xi_{1i}^{(N)}}{\sqrt{m_{2i}}} + \frac{\xi_{2i}^{(N)}}{m_{1i}} \to 0, N \to \infty$

(as usual, we assume that $\mathbf{p}_{ij} > 0$).

Example 3.1. Let $N \approx 10^6, n_1^c \approx n_2^c \approx 5 \cdot 10^5, m_{11} \approx m_{12} \approx m_{21} \approx m_{22} \approx 25 \cdot 10^4$. So $\mathbf{p}_1 = \mathbf{p}_2 = 1/2; \mathbf{p}_{11} = \mathbf{p}_{12} = \mathbf{p}_{21} = \mathbf{p}_{22} = 1/2$ (symmetric state). Suppose we have fluctuations (25) with $\xi_{1i}^{(N)} \approx \xi_{2i}^{(N)} \approx 1/2$. Then $\epsilon_{1i}^{(N)} \approx \epsilon_{2i}^{(N)} \approx 500$. So $n_{ij} = 24 \cdot 10^4 \pm 500$. Hence, the relative deviation $\frac{\epsilon_{ji}^{(N)}}{m_{ji}} = \frac{500}{25 \cdot 10^4} \approx 0.002$. Thus fluctuations of the relative magnitude $\approx 0,002$ produce the classical probabilistic rule.

It is evident that fluctuations of essentially larger magnitude

$$\epsilon_{1i}^{(N)} = 2\xi_{1i}^{(N)}(m_{1i})^{1/2}(m_{21})^{1/\alpha},\ \epsilon_{2i}^{(N)} = 2\xi_{2i}^{(N)}(m_{2i})^{1/2}(m_{1i})^{1/\beta}, \alpha, \beta > 2, \quad (26)$$

where $\{\xi_{1i}^{(N)}\}$ and $\{\xi_{2i}^{(N)}\}$ are bounded sequences ($N \to \infty$), also produce (for $\mathbf{p}_{ij} \neq 0$) the classical probability rule.

Example 3.2. Let all numbers $N, \ldots, m_{ij}$ be the same as in Example 3.1 and let deviations have behaviour (26) with $\alpha = \beta = 4$. Here the relative deviation $\frac{\xi_{ij}^{(N)}}{mij} \approx 0,045$.

4 Classical, quantum and 'superquantum' physics

In this section we find relations between different classes of physical experiments. First we consider so called classical and quantum experiments. Classical experiments produce the classical probabilistic rule (Bayes' formula). Therefore the corresponding ensemble fluctuations have the asymptotic $\delta_i^{(N)} \to 0, N \to \infty$.

Nevertheless, we cannot say that classical measurements give just a subclass of quantum measurements. In the classical domain we have no symmetric relations $\mathbf{p}_{11} = \mathbf{p}_{22}$ and $\mathbf{p}_{12} = \mathbf{p}_{21}$. This is the special condition which connects the preparation procedures $\mathcal{E}_1$ and $\mathcal{E}_2$. This relation is a peculiarity of quantum preparation/measurement procedures.

Experiments with nonclassical probabilistic rules are characterized by the condition $\delta_{\mathrm{i}}^{(\mathrm{N})} \not\to 0, \mathrm{N} \to \infty$. Quantum experiments give only a particular class of nonclassical experiments. Quantum experiments produce ensemble fluctuations of form (17), (18), where coefficients $\xi_{1i}^{(N)}$ and $\xi_{20}^{(N)}$ satisfy (19) and the orthogonality relation

$$\lim_{N\to\infty}(\xi_{11}^{(N)}+\xi_{21}^{(N)})+\lim_{N\to\infty}(\xi_{12}^{(N)}+\xi_{22}^{(N)})=0\ . \tag{27}$$

In particular, nonclassical domain contains (nonquantum) experiments which satisfy condition of boundedness (19), but not satisfy orthogonality relation (27). Here we have only the relation of quazi-orthogonality (22). In this case the matrix of probabilities is not double stochastic. The corresponding probabilistic rule has the form:

$$q_i=\mathbf{p}_1\mathbf{p}_{1i}+\mathbf{p}_2\mathbf{p}_{2i}+2\sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{1i}\mathbf{p}_{2i}}\cos\theta_i. \tag{28}$$

Here in general $\mathbf{p}_{11}+\mathbf{p}_{21}\neq 1, \mathbf{p}_{12}+\mathbf{p}_{22}\neq 1$.

We remark that, in fact, (28) and (22) imply that

$q_1=\mathbf{p}_1\mathbf{p}_{11}+\mathbf{p}_2\mathbf{p}_{21}+2\ \sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{11}\mathbf{p}_{21}}\cos\theta_1$;

$q_2=\mathbf{p}_1\mathbf{p}_{12}+\mathbf{p}_2\mathbf{p}_{22}-2\sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{11}\mathbf{p}_{21}}\cos\theta_1$.

5 Hyperbolic 'quantum' formalism

Let us consider ensembles $\mathrm{S}, \mathrm{T}_1, \mathrm{T}_2$ such that ensemble fluctuations have magnitudes (17), (18) where

$$|\xi_{1i}^{(N)}+\xi_{2i}^{(N)}|\geq 1+c, c>0, N\to\infty. \tag{29}$$

Here the coefficients $\lambda_i=\lim_{N\to\infty}(\xi_{1i}^{(N)}+\xi_{2i}^{(N)})$ can be represented in the form $\lambda_i=\mathrm{ch}\ \theta_\mathrm{i}, \mathrm{i}=1,2$. The corresponding probability rule is the following

$q_i=\mathbf{p}_1\mathbf{p}_{1i}+\mathbf{p}_2\mathbf{p}_{2i}+2\sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{1i}\mathbf{p}_{2i}}\ \mathrm{ch}\ \theta_\mathrm{i}, \mathrm{i}=1,2.$

The normalization $q_1+q_2=1$ gives the orthogonality relation:

$$\sqrt{\mathbf{p}_{11}\mathbf{p}_{21}}\ \mathrm{ch}\ \theta_1+\sqrt{\mathbf{p}_{12}\mathbf{p}_{22}}\ \mathrm{ch}\ \theta_2=0\ . \tag{30}$$

Thus $\mathrm{ch}\theta_2=-\mathrm{ch}\theta_1\sqrt{\frac{\mathbf{p}_{11}\mathbf{p}_{21}}{\mathbf{p}_{12}\mathbf{p}_{22}}}$ and, hence,

$q_2=\mathbf{p}_1\mathbf{p}_{12}+\mathbf{p}_2\mathbf{p}_{22}-2\ \sqrt{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_{11}\mathbf{p}_{21}}\ \mathrm{ch}\theta_1$.

Such a formalism can be called a *hyperbolic quantum formalism.* It describes a part of nonclassical reality which is not described by 'trigonometric quantum formalism'. Experiments (and preparation procedures $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$)

which produce hyperbolic quantum behaviour could be simulated on computer. On the other hand, at the moment we have no 'natural' physical phenomena which are described by the hyperbolic quantum formalism. 'Trigonometric quantum behaviour' corresponds to essentially better control of properties in the process of preparation than 'hyperbolic quantum behaviour'. Of course, the aim of any experimenter is to approach 'trigonometric behaviour'. However, in principle there might exist such natural phenomena that 'trigonometric quantum behaviour' could not be achieved. In any case even the possibility of computer simulation demonstrates that quantum mechanics (trigonometric) is not complete (in the sense that not all physical reality is described by the standard quantum formalism). [1]

Example 6.1. Let $\mathbf{p}_1 = \alpha, \mathbf{p}_2 = 1 - \alpha, \mathbf{p}_{11} = \ldots = \mathbf{p}_{22} = 1/2$. Then $q_1 = \frac{1}{2} + \sqrt{\alpha(1-\alpha)}\lambda_1, q_2 = \frac{1}{2} - \sqrt{\alpha(1-\alpha)}\lambda_1$.

If α is sufficiently small, then λ_1 can be, in principle, larger than 1:$\lambda_1 = \mathrm{ch}\theta$.

6 Quantum behaviour for macroscopic systems

Our analysis shows that 'quantum statistical behaviour' can be demonstrated by ensembles consisting of macroscopic systems; for example, balls having colours $C = c_1$, red, or c_2, blue, and weights $A = a_1 = 1$ or $a_2 = 2$. Suppose that additional filters $F_i, i = 1, 2$, produce fluctuations (17), (18), (27). Then, instead of classical Bayes' formula (1), we obtain quantum probability rule (2).

In the context of the statistical simulation of quantum statistical behaviour via fluctuations (17), (18) (with (27)) it would be useful to note that, in fact, we can choose constant coefficients $\xi_{\mathrm{ij}}^{(\mathrm{N})} = \xi_{\mathrm{ij}}$. Moreover, we have $\xi_{11} = -\xi_{12}$ and $\xi_{21} = -\xi_{22}$. The latter is a consequence of the general relations:

$$\frac{\xi_{11}^{(N)}}{\xi_{12}^{(N)}} \to -1, \quad \frac{\xi_{22}^{(N)}}{\xi_{21}^{(N)}} \to -1, N \to \infty. \tag{31}$$

Asymptotic (31) can be obtained from (17), (18):

[1] We can compare the hyperbolic quantum formalism with the hyperbolic geometry.

Proof. By (17) we have

$$(n_{11} - m_{11}) + (n_{12} - m_{12}) = 2\xi_{11}\sqrt{m_{11}m_{21}} + 2\xi_{12}\sqrt{m_{12}m_{22}}. \qquad (32)$$

The left hand side is equal to zero: $(n_{11} + n_{12}) - (m_{11} + m_{12}) = n_1^c - n_1^c = 0$ (as the ensemble T_1 has n_1^c elements). Hence, by (23) we get $\xi_{11} = -\xi_{12}\ \sqrt{\frac{m_{12}}{m_{21}}\frac{m_{22}}{m_{11}}} \to -\xi_{12}, N \to \infty$ (as $\mathbf{p}_{11} = \mathbf{p}_{22}$ and $\mathbf{p}_{12} = \mathbf{p}_{21}$). In the same way we obtain that $\xi_{21} = -\xi_{22}\ \sqrt{\frac{m_{12}}{m_{21}}\frac{m_{22}}{m_{11}}} \to -\xi_{22}, N \to \infty$.

Conclusion. *We demonstrated that so called quantum probabilistic rule has a natural explanation in the framework of ensemble fluctuations induced by preparation procedures. In particular, the quantum rule for probabilities (with nontrivial* $\cos\theta$*-factor) could be simulated for macroscopic physical systems via preparation procedures producing the special ensemble fluctuations.*

7 Appendix: correlations between preparation procedures

In this section we study the frequency meaning of the fact that in the quantum formalism the matrix of probabilities is double stochastic. We remark that this is a consequence of orthogonality of quantum states ϕ_1 and ϕ_2 corresponding to distinct values of a physical observable C. We have

$$\frac{\mathbf{p}_{11}}{\mathbf{p}_{12}} = \frac{\mathbf{p}_{22}}{\mathbf{p}_{21}} . \qquad (33)$$

Suppose that (a), see section 2, is the origin of quantum behaviour. Hence, all quantum features are induced by the impossibility to create new ensembles T_1 and T_2 without to change properties of quantum particles. Suppose that, for example, the preparation procedure $\mathcal{E}_1$ practically destroys the property $A = a_1$ (transforms this property into the property $A = a_2$). So $\mathbf{p}_{11} = 0$. As a consequence, the $\mathcal{E}_1$ makes the property $A = a_2$ dominating. So $\mathbf{p}_{12} \approx 1$. Then the preparation procedure $\mathcal{E}_2$ *must* practically destroy the property $A = a_2$ (transforms this property into the property $A = a_1$). So $\mathbf{p}_{22} \approx 0$. As a consequence, the $\mathcal{E}_2$ makes the property $A = a_1$ dominating. So $\mathbf{p}_{21} \approx 1$.

Frequency relation (23) can be represented in the following form:

$$\frac{m_{11}}{n_1^c} - \frac{m_{22}}{n_2^c} \approx 0, N \to \infty . \qquad (34)$$

We recall that the number of elements in the ensemble T_i is equal to n_i^c.

Thus

$$(\frac{n_{11} - m_{11}}{n_1^c}) - (\frac{n_{22} - m_{22}}{n_2^c}) \approx \frac{n_{11}}{n_1^c} - \frac{n_{22}}{n_2^c}. \tag{35}$$

This is nothing than the relation between fluctuations of property A under the transition from the ensemble S to ensembles T_1, T_2 and distribution of this property in the ensemble S.

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